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Fast, Robust, Iterative Riemann Solvers for the Shallow Water and Euler Equations

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Modern numerical methods for hyperbolic conservation laws require the solution of a large number of Riemann problems at each time step. Approximate Riemann solvers are used almost universally, due to their computational efficiency and robustness. With recent advances in computational power and algorithms, it is natural to ask whether iterative solvers might be competitive. Here we reconsider the use of iterative Riemann solvers for the shallow water equations (SWE) and Euler equations (EE), testing and comparing a wide range of initial guesses and iterative methods.

Shallow water equations

Without loss of generality, we consider a planar Riemann problem in which the initial discontinuity is aligned with the y-axis. Then the solution is that corresponding to the system of one-dimensional system of conservation laws

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = 0, \tag{1}$$

with $\mathbf{q} = \left[h, hu, hv\right]^T$ and $\mathbf{f}(\mathbf{q}) = \left[hu, hu^2 + \frac{1}{2}gh^2, huv\right]^T$. The solution for the Riemann problem associated to this system consists of four constant states separated by three waves, and it will be completely determined once we find the value of the depth h_* in the middle states. By means of the Rankine-Hugoniot condition and the Riemann invariants, we get that h_* is the unique root of the concave and increasing depth function $\phi^h(h; \mathbf{q}_\ell, \mathbf{q}_r) = f^h(h, h_\ell) + f^h(h, h_r) + u_r - u_\ell$, where

$$f^h(h, h_Z) = \begin{cases} 2(\sqrt{gh} - \sqrt{gh_Z}), & \text{if } h \le h_Z \text{ (rarefaction)}, \\ (h - h_Z)\sqrt{\frac{1}{2}g\left(\frac{h+h_Z}{hh_Z}\right)}, & \text{if } h > h_Z \text{ (shock)}. \end{cases} \tag{2}$$

Euler equations

We consider the Riemann problem for the one-dimensional Euler equations for an ideal gas, since the essential difficulties of solving a planar Riemann already arise in the 1D setting. Thus we consider (1) with $\mathbf{q} = \left[\rho, \rho u, E\right]^T$, $\mathbf{f}(\mathbf{q}) = \left[\rho u, \rho u^2 + p, u(E+p)\right]^T$, where $E = \rho\left(\frac{1}{2}u^2 + e\right)$ and $e = \frac{p}{(\gamma-1)p}$. The solution of this Riemann problem also consists of four constant states connected by three waves, and can be explicitly stated once the pressure p_* of the middle states is obtained. Through the isentropic relations and generalised Riemann invariants for rarefactions, and the Rankine-Hugoniot conditions for shocks, it can be shown that p_* is the unique root of the pressure function $\phi^p(p; \mathbf{w}_\ell, \mathbf{w}_\mathrm{r}) = f^p(p, \mathbf{w}_\ell) + f^p(p, \mathbf{w}_\mathrm{r}) + u_\mathrm{r} - u_\ell$. Here

$$f^p(p, \mathbf{w}_Z) = \begin{cases} \frac{2C_Z}{(\gamma-1)}\left[\left(\frac{p}{p_Z}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right] & \text{if } p \le p_Z (\text{ rarefaction }), \\ (p - p_Z)\left[\frac{C_Z}{p+B_Z}\right]^{\frac{1}{2}} & \text{if } p > p_Z (\text{ shock }), \end{cases} \quad \text{and} \quad C_Z = \frac{2}{(\gamma+1)\rho_Z}, \quad B_Z = \frac{(\gamma-1)}{(\gamma+1)}p_Z. \tag{3}$$

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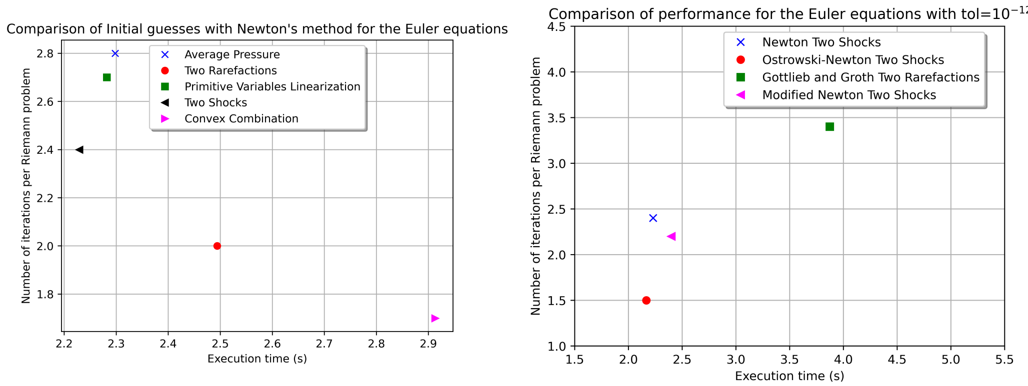

Figure 1: Performance of initial guesses and iterative algorithms for the Euler equations

Iterative Riemann solvers

Almost all the cost of computing exactly the solution for the Riemann problem for these systems consists in finding h_* and p_*. Therefore, our goal is to obtain an optimal root finding procedure for ϕ^h and ϕ^d, for which we require initial guesses and iterative algorithms. A number of initial guesses and iterative solvers have been proposed these systems, focusing mostly on the Euler equations; see for example [3, 4] and references therein. Due to the similar characteristic structure of the shallow water system, these ideas can be applied immediately to either system, and we will discuss them together.

We consider several options for the initial guess, including: (i) the two-rarefaction approximation; (ii) for the EEs only, a linearization in primitive variables; (iii) a two-shock approximation based on the aforementioned linearization for the EEs or on an averaged depth for the SWEs; (iv) a highly accurate but expensive initial guess we have designed based on a convex combination of certain bounds on the root of ϕ; (v) for the SWEs only, an initial guess suggested in [1].

We also consider a wide range of iterative algorithms. We prove for both systems that Newton's method with a simple fix converges to the root of ϕ independently of the initial guess. We test: (i) Newton's method with this fix; (ii) a modified Newton's method with a rate of convergence of $\sqrt{2}+1$; (iii) Ostrowski's third order multipoint method; (iv) a combination of Ostrowski's method with Newton's method that preserves the high order of convergence without losing robustness; (v) an adaptation of a method to iteratively estimate the characteristic wave speeds [2]; (vi) the method of van Leer; (vii) the method of Gottlieb, and Groth (for the last two methods see [3]).

To test the performance of the initial guesses and iterative algorithms, we take a random set of 10^7 Riemann problems. As proposed in [3], for each 10 Riemann problems, 2 involve strong waves and 8 involve only weak waves. In all cases, we perform a simple *a priori* check for a two-rarefaction solution, and use the corresponding explicit solution in that case. All initial guesses are tested with Newton's method to determine not only their accuracy but their effect when used with an iterative solver. For the SWEs the two-shock approximation performs noticeably better than the other initial guesses. These results are similar to those obtained for the EEs and perhaps not surprising given that we apply the exact solution for any two-rarefaction cases before iterating. Motivated by these results, to test the iterative solvers the two-shock initial guess is used when possible. For the SWEs we obtain that the modified Newton's method is slightly faster, however the difference is not great enough to make it preferable to the robust Newton's method with the convergence fix. In the case of the EEs, the Ostrowski-Newton combination converges around 5% faster than Newton's method when the prescribed tolerance is 10^{-12} and around 10% slower when it is 10^{-6}. This indicates that for these kind of problems low-order and low-cost methods are more convenient. However, higher order methods like Ostrowski's might be the best approach when extreme accuracy is required or the complexity of the system of algebraic equations increases.

References

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