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# Convergence and error estimates of the Godunov Method for Multidimensional Compressible Euler Equations

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It is well-known that for multidimensional compressible Euler equations there may exist infinitely many weak entropy solutions [2, 5]. In order to describe oscillations arising in the limits of singular perturbations of hyperbolic conservation laws, a more generalised solution, i.e. measure-valued (MV) solution, was proposed by DiPerna [4]. In this talk, we use the recently developed concept of *dissipative measure-valued (DMV) solution* [1] and an elegant tool – *the relative energy* [3] and show the convergence and error estimates of the first-order finite volume method based on the exact Riemann solver for the compressible Euler equations. For more details about the DMV solution and the relative energy, we refer to the works [6, 7] and books [8, 9].

Specifically, assuming that the numerical densities are uniformly bounded from below by a positive constant and that the numerical energies are uniformly bounded from above we

- derive the entropy inequality and weak BV estimates;
- prove that this hypothesis is equivalent to the strict uniform convexity of the mathematical entropy;
- derive the consistency formulation of the Godunov method;
- show the weak* convergence of numerical solutions to a DMV solution, see [10].

  If the limit of a subsequence of numerical solutions is a weak or $C^1$ entropy solution, then the convergence is also strong; if the Euler system admits a strong solution, then the numerical solutions strongly converge to the strong solution as long as the latter exists.

Further, under the assumption of the existence of the strong solution we

- obtain a convergence rate of 1/2 for the relative energy in the $L^1$-norm, i.e. a convergence rate of 1/4 for the $L^2$-error of the numerical solution;
- obtain the first order convergence rate for the relative energy in the $L^1$-norm if the total variations of numerical solutions are uniformly bounded. It means that, the numerical solutions converge with the convergence rate of 1/2 in the $L^2$-norm, see [11].

Experiments have been performed to confirm the theoretical results, including 2D spiral, Kelvin-Helmholtz, Richtmyer-Meshkov problems for the convergence part, and several 1D and 2D Riemann problems for error estimates part.

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