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Balance Laws with Singular Source Term and Applications to Fluid Dynamics

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Conservation laws in one space dimension, i.e., systems of partial differential equations in conservative form of the type

$$\partial_t u + \partial_x f(u) = 0 \qquad t \geq 0,\ x \in \mathbb{R}\,, \tag{1}$$

allow to describe, for instance, the movement of a fluid along a rectilinear pipe with constant section. Assume that at a point $\bar{x}$ the pipe's direction or its section changes. Then, equation (1) can be used, separately, where $x < \bar{x}$ and where $x > \bar{x}$. At the point $\bar{x}$, on the basis of physical considerations, a further condition is necessary to prescribe the possible defect in the conservation of the various variables. Typically, such a condition is written as

$$\Psi\left(z^+, u(t,\bar{x}+), z^-, u(t,\bar{x}-)\right) = 0 \quad \text{for a.e. } t > 0, \tag{2}$$

where z^+ and z^-, identify the physical parameters that change across $\bar{x}$. Alternatively, introducing a function Ξ to measure the defect in the conservation of the u variable, under suitable assumptions, (2) can be rewritten as

$$f\left(u(t,\bar{x}+)\right) - f\left(u(t,\bar{x}-)\right) = \Xi\left(z^+, z^-, u(t,\bar{x}-)\right) \quad \text{for a.e. } t > 0. \tag{3}$$

It is then natural to tackle the resulting Riemann Problem, that is, the Cauchy Problem consisting of (1)–(3) with an initial datum attaining two values, one for $x < 0$ and another one for $x > 0$. The finite propagation speed, intrinsic to (1), allows then to extend the whole construction to any finite number of points $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_k$, essentially solving the Cauchy Problem for the balance law

$$\begin{cases} \partial_t u + \partial_x f(u) = \sum_{i=1}^{k-1} \Xi\left(\zeta_k(\bar{x}_i+), \zeta_k(\bar{x}_i-), u(t,\bar{x}_i-)\right)\ \delta_{\bar{x}_i} \\ u(0,x) = u_o(x), \end{cases} \tag{4}$$

where $\delta_{\bar{x}_i}$ denotes the Dirac measure at $\bar{x}_i$ and ζ_k is the piecewise constant function attaining the $k+1$ constant values $z_0, z_1, \ldots, z_k$ on the intervals $]-\infty, \bar{x}_1[,\ ]\bar{x}_1, \bar{x}_2[, \ldots,]\bar{x}_k, +\infty[$.

We provide a detailed description of the rigorous limit $k \to +\infty$ of (4), covering its extension to the case of a general $\mathbf{BV}$ function $\zeta \in \mathbb{R}^p$ under the non resonance condition, i.e., we require that all eigenvalues of (1) be separated from 0.

In particular, solutions to (4) with initial datum u_o are shown to converge as $k \to +\infty$ to solutions to

$$\begin{cases} \partial_t u + \partial_x f(u) = \sum_{x\in\mathcal{I}} \Xi\left(\zeta(x+), \zeta(x-), u(t,x-)\right)\delta_x + D^+_{v(x)}\Xi\left(\zeta(x), \zeta(x), u(t,x)\right)\|\mu\| \\ u(0,x) = u_o(x)\,. \end{cases} \tag{5}$$

Since $\zeta \in \mathbf{BV}(\mathbb{R};\mathbb{R}^p)$, the right and left limits $\zeta(x+)$ and $\zeta(x-)$ are well defined and the distributional derivative $D\zeta$ can be split in a discrete part and a continuous one, which may contain a Cantor part:

$$D\zeta = \sum_{x\in\mathcal{I}} \left(\zeta(x+) - \zeta(x-)\right)\ \delta_x + v\,\|\mu\|\,, \tag{6}$$

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where the function v is Borel measurable with norm 1, μ is the non atomic part of $D\zeta$ and $\mathcal{I}$ is the set of jump points in ζ. In (5) we also used the (one sided) directional derivative

$$D_v^+ \Xi(z, z, u) = \lim_{t \to 0+} \frac{\Xi(z + t\, v, z, u) - \Xi(z, z, u)}{t}\,. \tag{7}$$

Indeed, one of our motivating examples, namely the case of a curved pipe, leads to a function Ξ that admits directional derivatives but is not differentiable.

On the other hand, as soon as Ξ is differentiable with respect to its first argument, the right hand side in (5) can be slightly simplified, to

$$D_{v(x)}^+ \Xi(a, a, u)\, \|\mu\| = D_1\Xi(a, a, u)\, v(x)\, \|\mu\| = D_1\Xi(a, a, u)\, \mu\,. \tag{8}$$

The existence of solutions to (5) is achieved sequentially combining wave front tracking with the approximation of the equation, in particular of the map ζ. A key role is played by a very careful choice of these approximations.

The above general procedure, when applied to the case of a curved pipe with constant section, amounts to justify the role of the pipe's curvature on the fluid flow inside the pipe. Indeed, if x is the abscissa along the pipe and $\Gamma = \Gamma(x)$ describes the pipe's shape, then the pipe's local direction that enters the equation for fluid flow is $\zeta(x) = \Gamma'(x)$. Problem (4) then corresponds to a piecewise linear pipe and (the second component of) (3) describes the change in the fluid linear momentum at a kink sited at $\bar{x}$. Assuming that the lack in the conservation of linear momentum depends on the angle in the pipe at $\bar{x}$, i.e., $\Xi(z^+, z^-, u) = K(\|z^+ - z^-\|, u)$ as in [2], we show that, in the smooth pipe limit, the variation in the fluid momentum depends on the pipe's curvature Γ''.

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