An asymptotic preserving discretization scheme for gas transport in pipe networks

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We consider the gas transport in long pipes and one-dimensional pipe networks. The flow on each pipe, identified with the interval $(0,\ell)$, is modeled by the barotropic Euler equations which after transformation and rescaling are given by

$$a\partial_\tau \rho + \partial_x(a\rho w) = 0, \tag{1}$$

$$\varepsilon^2 \partial_\tau w + \partial_x\big(\frac{\varepsilon^2}{2} w^2 + P'(\rho)\big) = -\gamma |w| w. \tag{2}$$

Here a is the cross-sectional area of the pipe, ρ the gas density, τ, w and γ the rescaled time, velocity and friction coefficient, $P(\rho)$ the pressure potential, and ε a scaling parameter, proportional to the Mach number. We are interested in the low-Mach resp. high friction limit $\varepsilon \to 0$ which corresponds to the typical setting of long length and time scales of practical relevance; see [1] for details. In that case, one can expect smooth solutions bounded away from vacuum, which we assume in the following. By formally setting $\varepsilon = 0$ in (1)–(2) one obtains a parabolic model for gas transport which is widely used in practice [2].

Abstract Hamiltonian formulation. We call ρ, w the *state variables* and denote by $h := \frac{\varepsilon^2}{2} w^2 + P'(\rho)$, $m := a\rho w$ the *co-state variables* of the above system, i.e., total specific enthalpy and mass flux. These are linked via $ah = \frac{\delta \mathcal{H}}{\delta \rho}$ and $\varepsilon^2 m = \frac{\delta \mathcal{H}}{\delta w}$ to the variational derivatives of the associated energy functional

$$\mathcal{H}(\rho, w) := \int_0^\ell a\big(\frac{\varepsilon^2}{2}\rho w^2 + P(\rho)\big)\, dx. \tag{3}$$

These new variables allow us to derive the following weak form of the problem: By multiplying (1)–(2) with suitable test functions, integrating over the pipe and apply integration-by-parts in the second equation, we obtain

$$(a\partial_\tau \rho(\tau), q) + (\partial_x m(\tau), q) = 0, \tag{4}$$

$$(\varepsilon^2 \partial_\tau w(\tau), r) - (h(\tau), \partial_x r) = -(\gamma |w(\tau)| w(\tau), r) - h(\tau) r\big|_0^\ell, \tag{5}$$

with $(u,v) = \int_0^\ell uv\, dx$ denoting the standard L^2-scalar product and $\tau > 0$. Any smooth solution of (1)–(2) is characterized by these variational identities. The system (4)–(5) can further be written as an abstract dissipative Hamiltonian system of the form

$$\mathcal{C}\partial_\tau \boldsymbol{u} + \big(\mathcal{J} + \mathcal{R}(\boldsymbol{u})\big)\boldsymbol{z}(\boldsymbol{u}) = \mathcal{B}_\partial \boldsymbol{z}(\boldsymbol{u}), \quad \boldsymbol{z}(\boldsymbol{u}) = \mathcal{C}^{-1}\mathcal{H}'(\boldsymbol{u}) \tag{6}$$

with $\boldsymbol{u} = (\rho, w)$ and $\boldsymbol{z}(\boldsymbol{u}) = (h, m)$ denoting the state and co-state variables, and with appropriate operators $\mathcal{C}$, $\mathcal{J}$, $\mathcal{R}$, $\mathcal{B}_\partial$, where $\mathcal{C}$ is positive definite, $\mathcal{J}$ skew-symmetric, $\mathcal{R}(\boldsymbol{u})$ positive, and $\mathcal{B}_\partial$ a boundary operator. From these properties and the abstract form of the problem, we immediately deduce the following energy identity or inequality for smooth solutions of (6)

$$\frac{d}{d\tau}\mathcal{H}(\boldsymbol{u}) = \langle \partial_\tau \boldsymbol{u}, \mathcal{H}'(\boldsymbol{u})\rangle = -\langle \mathcal{R}(\boldsymbol{u})\boldsymbol{z}(\boldsymbol{u}), \boldsymbol{z}(\boldsymbol{u})\rangle + \langle \mathcal{B}_\partial \boldsymbol{z}(\boldsymbol{u}), \boldsymbol{z}(\boldsymbol{u})\rangle \le \langle \mathcal{B}_\partial \boldsymbol{z}(\boldsymbol{u}), \boldsymbol{z}(\boldsymbol{u})), \tag{7}$$

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where $\langle \cdot, \cdot \rangle$ denotes the duality product. This relation shows that the change in the system energy over time is caused only by dissipation due to friction at pipe walls and by flux over the boundary, and hence the system is passive.

Asymptotic stability. We now study the stability of solutions $\boldsymbol{u} = (\rho, w),\ \hat{\boldsymbol{u}} = (\hat{\rho}, \hat{w})$ to (1)–(2) for different scaling parameters $\varepsilon,\ \hat{\varepsilon}$. To do so, we use the concept of *relative energy* [3], which is defined by

$$\mathcal{H}(\boldsymbol{u}|\hat{\boldsymbol{u}}) = \mathcal{H}(\boldsymbol{u}) - \mathcal{H}(\hat{\boldsymbol{u}}) - \langle \mathcal{H}'(\hat{\boldsymbol{u}}), \boldsymbol{u} - \hat{\boldsymbol{u}} \rangle. \tag{8}$$

For appropriately bounded subsonic states, the relative energy introduces a distance measure which is equivalent to an ε-weighted L^2-norm. If the pressure potential is smooth and strictly convex, the flow subsonic and the solutions sufficiently smooth and uniformly bounded, i.e., $0 < \underline{\rho} \le \rho, \hat{\rho} \le \bar{\rho},\ -\bar{w} \le w, \hat{w} \le \bar{w}$, as well as $0 \le \varepsilon, \hat{\varepsilon} \le \bar{\varepsilon}$, we can further show that

$$\|\rho(\tau) - \hat{\rho}(\tau)\|^2_{L^2(0,\ell)} + \varepsilon^2 \|w(\tau) - \hat{w}(\tau)\|^2_{L^2(0,\ell)} + \int_0^\tau \|w(s) - \hat{w}(s)\|^3_{L^3(0,\ell)} \le C e^{c\tau} |\varepsilon^2 - \hat{\varepsilon}^2| \tag{9}$$

with constants c, C only depending on bounds for the solutions and parameters; see [4] for more details. Let us note that this result holds in particular for $\hat{\varepsilon} = 0$, and thus yields asymptotic stability towards the parabolic limit problem.

Asymptotic preserving discretization. Based on the variational formulation (4)–(5) we propose a structure preserving discretization scheme by Galerkin projections. We use a mixed finite element method in space, approximating ρ by a piecewise constant approximation ρ_h, and m by a piecewise linear, globally continuous approximation m_h, together with an implicit Euler time discretization. This corresponds to a standard approximation for related linear wave propagation problems [5]. Let us note that $w_h = w(\rho_h, m_h)$ and $h_h = h(\rho_h, m_h)$ are defined explicitly as functions of the discrete approximations $\rho_h,\ m_h$ for density and mass flux. By formally setting $\varepsilon = 0$ we obtain a viable numerical method for the parabolic limit problem, i.e., the scheme is *asymptotic preserving*. Since the basic Hamiltonian structure of the problem is inherited by the discrete approximation scheme, also the stability analysis via relative energy estimates directly transfers. Similar techniques for the analysis of numerical methods have been employed, e.g., in [6] for the Navier-Stokes equations. Under certain regularity assumptions on the solution ρ and w, we can even derive quantitative convergence rates that are uniform in the asymptotic parameter ε, i.e.,

$$\|\rho(\tau^n) - \rho_h^n\|^2_{L^2(0,\ell)} + \varepsilon^2 \|m(\tau^n) - m_h^n\|^2_{L^2(0,\ell)} + \sum_{k=1}^n \Delta\tau \|m(\tau^k) - m_h^k\|^3_{L^3(0,\ell)} \le C(\Delta\tau^2 + h^2) \tag{10}$$

with constant C being independent of ε; see [7] for more details. The error estimate also remains valid in the limit of $\varepsilon = 0$.

Extension to networks. Due to the use of a variational framework, our results generalise quite easily to pipe networks, which are described by finite, directed and connected graphs. On every edge of the graph, corresponding to a pipe of the network, the equations (1)–(2) are assumed to hold. Additional coupling conditions at pipe junctions allow to guarantee conservation of mass and energy [8]. The weak formulation of this problem again leads to a system of the abstract form (6), such that the stability results derived for a single pipe carry over to the network almost immediately. Also the numerical scheme as well as its stability and convergence analysis via relative energy estimates carry over almost verbatim.

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