XVIII International Conference on Hyperbolic Problems
Theory, Numerics, Applications
Málaga, June 20-24, 2022

Euler equations in fluid dynamics: Good and bad news

Eduard Feireisl *

We discuss some recent results concerning the Euler equations of compressible fluid flow. For the sake of simplicity, we focus on the isentropic case:

$$\partial_t \varrho + \operatorname{div}(\varrho \vec{u}) = 0,$$
$$\partial_t (\varrho \vec{u}) + \operatorname{div}(\varrho \vec{u} \otimes \vec{u}) + \nabla p(\varrho) = 0,$$

where the pressure is related to the density by the isentropic equations of state,

$$p(\varrho) = a\varrho^\gamma,\ a > 0,\ \gamma > 1.$$

0.1 Continuity in time of weak solutions

There is a number of negative results concerning the well posedness of the Euler system in the class of weak solutions. Most of them based on the recent development of the method of *convex integration* in the framework of fluid mechanics due to the series of works of De Lellis and Székelihydi and others. Here, we focus on the continuity in time of the weak solutions necessary for a meaningful definition of the initial data. For definiteness, we consider the space periodic boundary conditions,

$$x \in \Omega = \mathbb{T}^d,\ d = 2, 3.$$

The results however hold for the physically more realistic impermeability condition

$$\vec{u} \cdot \vec{n}|_{\partial\Omega} = 0,$$

where $\Omega \subset R^d$ is the physical domain occupied by the fluid.

The weak solutions of the Euler system satisfy

$$[\varrho, (\varrho\vec{u})] \in C_{\rm weak}([0, T]; L^\gamma(\Omega)) \times L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)).$$

This results is sharp in the class of weak solutions, meaning for any initial data

$$\varrho(0, \cdot) \text{ Riemann integrable in } \Omega,$$
$$\varrho\vec{u}(0, \cdot),\ \operatorname{div}(\varrho\vec{u})(0, \cdot) \text{ Riemann integrable in } \Omega,\ \varrho\vec{u}(0, \cdot) \cdot \vec{n}|_{\partial\Omega} = 0,$$

and any countable set of times $\{\tau_i\}_{i=1}^\infty \subset (0, T)$, the Euler system admits infinitely many weak solutions that are not strongly continuous at any τ_i. In addition, these solutions satisfy the energy inequality

$$\frac{\rm d}{\rm dt} \int_\Omega \left[\frac{1}{2} \varrho |\vec{u}|^2 + P(\varrho) \right] \ {\rm d}x \leq 0,\ P(\varrho) = \frac{a}{\gamma - 1} \varrho^\gamma,$$

see [1].

*Institute of Mathematics, Czech Academy of Sciences, Prague. Email: feireisl@math.cas.cz

0.2 Measurable semigroup selection

Despite the number of ill–posedness results, there is a semigroup selection of suitable *generalized* solutions to the Euler system. The generalized or *dissipative* solutions satisfy the following system of equations in the sense of distributions:

$$\partial_t \varrho + \mathrm{div}(\vec{m}) = 0,$$
$$\partial_t \vec{m} + \mathrm{div}\left(\frac{\vec{m} \otimes \vec{m}}{\varrho}\right) + \nabla p(\varrho) = -\mathrm{div}\, \mathcal{R},$$

together with the energy inequality

$$\int_\Omega \left[\frac{1}{2}\frac{|\vec{m}|^2}{\varrho} + P(\varrho)\right](\tau, \cdot)\mathrm{d}x + \mathfrak{E}(\tau, \cdot) \leq \int_\Omega \left[\frac{1}{2}\frac{|\vec{m}|^2}{\varrho} + P(\varrho)\right](0, \cdot)\mathrm{d}x,$$

where $\mathcal{R} \in R^{d\times d}_{\mathrm{sym}}$, and $\mathcal{R} \geq 0$,

$$\int_\Omega \mathrm{trace}[\mathcal{R}]\, \mathrm{d}x \leq C\, \mathfrak{E}.$$

The Euler system admits a measurable semiflow selection in the class of *dissipative* solutions. Specifically, we consider three state variables $(\varrho, \vec{m})$ together with the augmented energy

$$\mathcal{E} = \int_\Omega \left[\frac{1}{2}\frac{|\vec{m}|^2}{\varrho} + P(\varrho)\right] \mathrm{d}x + \mathfrak{E}$$

as functions of the time $t \in [0, \infty)$. Then there is a Borel measurable mapping

$$\mathcal{S} : t \geq 0,\ (\varrho_0, \vec{m}_0, \mathcal{E}_0) \mapsto (\varrho(t, \cdot), \vec{m}(t, \cdot), \mathcal{E}(t))(\varrho_0, \vec{m}_0, \mathcal{E}_0),$$

such that

$$\mathcal{S}[0; \varrho_0, \vec{m}_0, \mathcal{E}_0] = (\varrho_0, \vec{m}_0, \mathcal{E}_0),\ \mathcal{S}[t + s; \varrho_0, \vec{m}_0, \mathcal{E}_0] = \mathcal{S}\Big[s; \mathcal{S}[t; \varrho_0, \vec{m}_0, \mathcal{E}_0]\Big],\ s \geq 0.$$

For more details, see [2].

0.3 Euler system as vanishing viscosity limit, relevance to models of turbulence

Consider the Euler system

$$\mathrm{d}\varrho + \mathrm{div}(\varrho\vec{u})\mathrm{d}t = 0,$$
$$\mathrm{d}(\varrho\vec{u}) + \mathrm{div}(\varrho\vec{u} \otimes \vec{u})\mathrm{d}t + \nabla p(\varrho)\mathrm{d}t = \vec{F}(\varrho, \vec{m})\mathrm{d}W,$$

driven by a stochastic forcing. We address the question, to which extent solutions of the Euler system can be “statistically equivalent” to the inviscid limit of the Navier–Stokes system

$$\partial_t \varrho + \mathrm{div}(\varrho\vec{u}) = 0,$$
$$\partial_t(\varrho\vec{u}) + \mathrm{div}(\varrho\vec{u} \otimes \vec{u}) + \nabla p(\varrho) = \mu\Delta\vec{u} + \lambda\nabla\mathrm{div}\vec{u},\ \mu, \lambda \searrow 0$$

past a convex obstacle in R^3. See [3] for details.

Acknowledgements

This research has been partially supported by the Czech Sciences Foundation (GAČR), Grant Agreement 21–02411S. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

References

[1] A. Abbatiello and E. Feireisl. On strong continuity of weak solutions to the compressible Euler system. *J. Nonlinear Sci.*, **31**(2):Paper No. 33, 16, 2021.

[2] D. Breit, E. Feireisl, and M. Hofmanová. Solution semiflow to the isentropic Euler system. *Arch. Ration. Mech. Anal.*, **235**(1):167–194, 2020.

[3] E. Feireisl and M. Hofmanová. Randomness in compressible fluid flows past an obstacle. *Journal of Statistical Physics*, **186**:32–, 2022.