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Experiments with Entropy Conservative Flux Functions

K. Bahuguna [*], R. Kolluru [†], S. V. Raghurama Rao [‡]

Developing suitable numerical diffusion appropriate to entropy conservation is currently the focus of research in algorithm development, after decades of sophisticated developments made in shock capturing. Tadmor [1] introduced an appropriate mathematical framework for this purpose, followed by different variations by Roe [2] and a few others. All these schemes have a heavy dependence on the eigen-structure of the hyperbolic system. We report further experiments with alternative entropy conservative flux functions, coupled with exact shock capturing in a simple central discretization framework.

We start with the condition introduced by Tadmor an entropy conservative flux F^c.

(1) $$\Delta V \cdot F^c = \Delta \psi$$

where V is the entropy variable vector and ψ is entropy flux function. A family of entropy conservative fluxes can be constructed by simply writing ψ as a function of entropy variables in the above condition. Then we obtain

$$\Delta V \cdot F^c = \Delta \psi(V) \quad , \quad \psi(V) = v_2(-v_3)^{\frac{-\gamma}{\gamma-1}} \exp\left(\frac{-\gamma}{\gamma-1} + v_1^2 - \frac{v_2^2}{2v_3}\right)$$

There are multiple ways in which $\Delta\psi$ can be split and they are used to compute different entropy conservative fluxes. One variation for the entropy conservative flux (EC1) is given by the following expressions.

(2) $$\begin{aligned} F_1^c =& M_A(v_2) M_A((-v_3)^n) \exp(M_{EX}(k)) \\ F_2^c =& F_1^c \left(\frac{M_A(v_2)}{M_{HR}(-v_3)}\right) + M_A(\exp(k)(-v_3)^n) \\ F_3^c =& F_1^c \left(\frac{1}{2} \frac{M_A(v_2^2)}{M_A(-v_3) M_{HR}(-v_3)}\right) - n M_A(\exp(k)) M_A(v_2) \left(M_{GL}^n(-v_3)\right)^{n-1} \end{aligned}$$

Here M_A is the arithmetic mean, M_{GL} is generalized log mean and M_{EX} is exponential mean. Numerically stable approximations of M_{GL} and M_{EX} are used. The above flux preserves steady contact discontinuities exactly, a property not shared by other variations.

For capturing discontinuities, the framework of simple and robust central schemes given by Kolluru *et al.*[3] is used. One of them is the scalar diffusion based on generalized Riemann invariants. This scheme can capture steady contact discontuities exactly. An entropy stable scheme is then obtained by

$$F = F^c + F_R^d, \quad F_R^d = -\frac{1}{2}\alpha_I \Delta U$$

Where $\alpha_I = max(|u_L|, |u_R|)$ and F^c is entropy conservative flux given by (2). Above diffusion is based on the semi-discrete entropy conservation law and thus adds diffusion consistently. Another additional diffusion at shocks is taken based on Rankine-Hugoniot jump condition, as in [3]. The entropy stable flux then is given by

(3) $$F = F^c + F_M^d, \qquad F_M^d = -\frac{1}{2}|s_{j+1/2}|\Delta U$$

[*] PhD student, Indian Institute of Science. Email: kunalb@iisc.ac.in
[†] Post-doctoral fellow Indian Institute of Science. Email: kollurur@iisc.ac.in
[‡] Associate Professor, Indian Institute of Science. Email: raghu@iisc.ac.in

A switch based on the gradient of entropy is used to detect shocks and MOVERS+ type diffusion is used only at shocks. Final flux is given by

$$F = F^c + (1-\phi)F_R^d + \phi F_M^d, \quad \phi = \begin{cases} 1 & \text{if shock present} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Numerical experiments are performed on a typical benchmark one dimensional test cases, as shown in figure 1-8. The scheme captures both shocks and contact discontinuities with low diffusion and captures steady contact discontinuities exactly. Strong shocks are captured without any oscillations or anomalies.

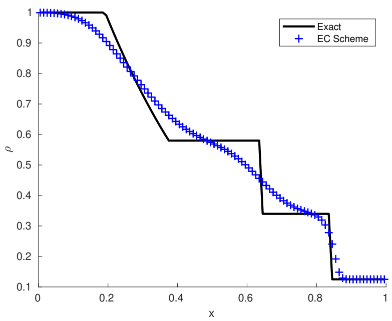

Figure 1: Sod's shock tube test case

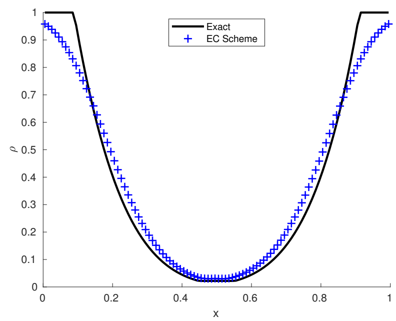

Figure 2: Double rarefraction test case

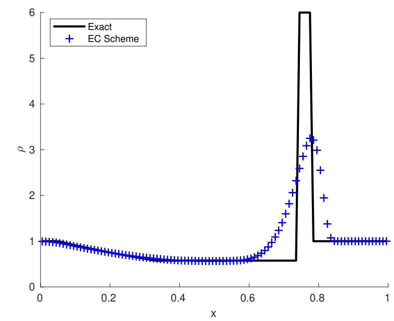

Figure 3: Strong shock tube test case

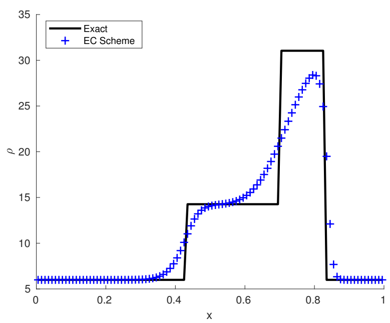

Figure 4: Two shocks separated by a contact discontinuity

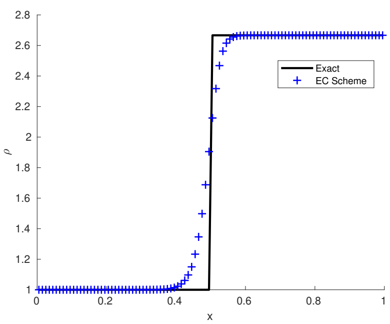

Figure 5: Steady shock test case

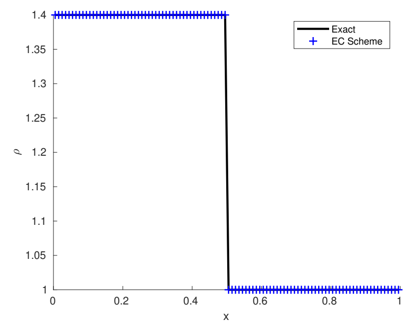

Figure 6: Steady contact discontinuity

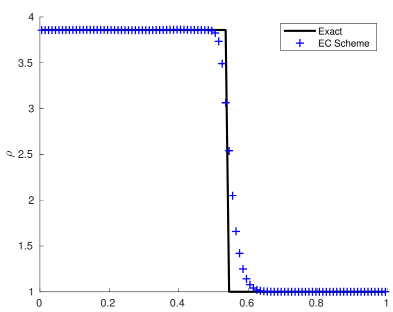

Figure 7: Slowly moving shock captured without oscillations

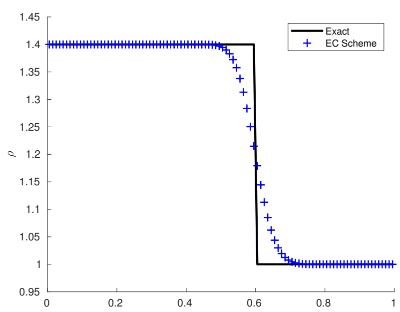

Figure 8: Slowly moving contact test case

References

[1] E. Tadmor. The numerical viscosity of entropy stable schemes for systems of conservation laws. *Mathematics of Computation*, 49 : 91-103 , 1987.

[2] P.L. Roe. Affordable, entropy-consistent Euler flux functions I. In *Eleventh International Conference on Hyperbolic Problems: Theory, Numerics, Applications*, Lyon, 2006.

[3] R. Kolluru, N.V. Raghavendra, S.V. Raghurama Rao and G.N. Sekhar, Simple and robust contact-discontinuity capturing schemes for high speed compressible flows. *Applied Mathematics and Computation*, 414 , 2021.