XVIII International Conference on Hyperbolic Problems
Theory, Numerics, Applications
Málaga, June 20-24, 2022

Traffic Flow Models with Noise

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Hyperbolic PDEs can be used to describe the macroscopic dynamics of traffic flow. Equilibrium models (also called first order models) are scalar conservation laws expressing mass conservation [8, 10]

$$\rho_t + (\rho v)_x = 0 \tag{1}$$

where ρ denotes the flow density, and v the flow velocity. The closure relation

$$v = V(\rho)$$

is called the *Fundamental Diagram* (FD), and describes a velocity that instantaneously adjusts itself to flow density. The FD is largely approximated from observation, and in general $V(\rho)$ is a non-increasing function of ρ.

Dynamic generalizations (also called second order models) have been proposed by numerous authors [1, 5, 11, 12], and have the general form

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ v_t + (v - g(\rho))\, v_x &= \frac{V(\rho) - v}{\tau^v} \end{aligned} \tag{2}$$

where the velocity $v = v(x, t)$ does not adjust instantaneously to traffic density, but instead is governed by an equation that describes rules for acceleration such as a reaction to changes in local flow conditions (e.g. traffic slowing down ahead) and relaxation towards an optimal equilibrium velocity $V(\rho)$. In analogy to fluid dynamics, $g(\rho)$ is sometimes referred to as traffic 'sound speed' as it governs the nonlinear propagation of small disturbances.

Driver behavior, however, differs among drivers and over time; this variability is not captured by deterministic models. Indeed, real data suggests that while one may identify 'mean' driver behavior, in general variability in driving style results in some distribution around the mean [4, 7, 9]. To model driver variability, we introduce a (small) driver-related parameter z that describes deviation from the mean and create a family of fundamental diagrams

$$V(\rho, z).$$

Figure 1 shows a prototype family of fundamental diagrams inspired by real data illustrating a typical FD and velocity variance. (For related work see [2, 4, 7, 9].) The dynamics of z are governed by a stochastic advection diffusion equation

$$z_t + V(\rho, z)\, z_x = \kappa z_{xx} - \frac{z}{\tau^z} + \eta\xi \tag{3}$$

where ξ is a Gaussian white noise varying in both space and time, $V(\rho, z)\, z_x$ is advection at the speed of traffic, κz_{xx} is related to autocorrelation of noise in space, and $-z/\tau^z$ is a relaxation to the mean preventing drift in behavior over time.

The resulting stochastic models adhere to accepted principles for traffic modeling (for example those in [1, 3]) and are capable of reproducing a richer set of traffic flow phenomena. Most notably, they illustrate that small perturbations may grow into large coherent wave structures including the formation of jams and emergence of stop-and-go flow patterns (Figure 2).

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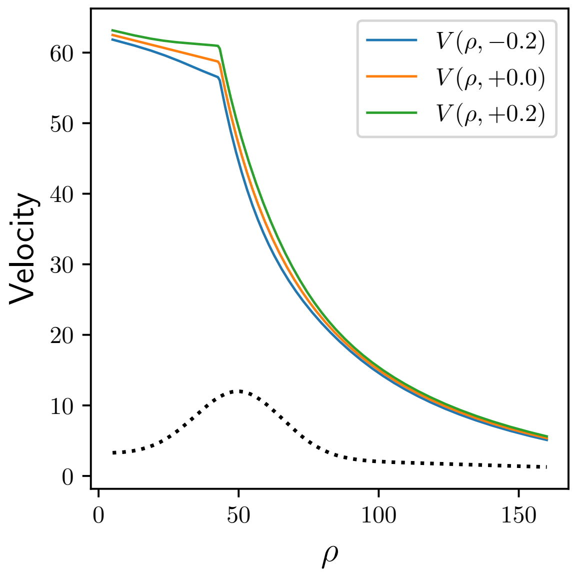

Figure 1: A fundamental diagram of the form $V(\rho, z) = V_0(\rho) + zV_1(\rho)$ where $V_0(\rho)$ (orange curve) and $V_1(\rho)$ (dotted black curve) are inspired by real data.

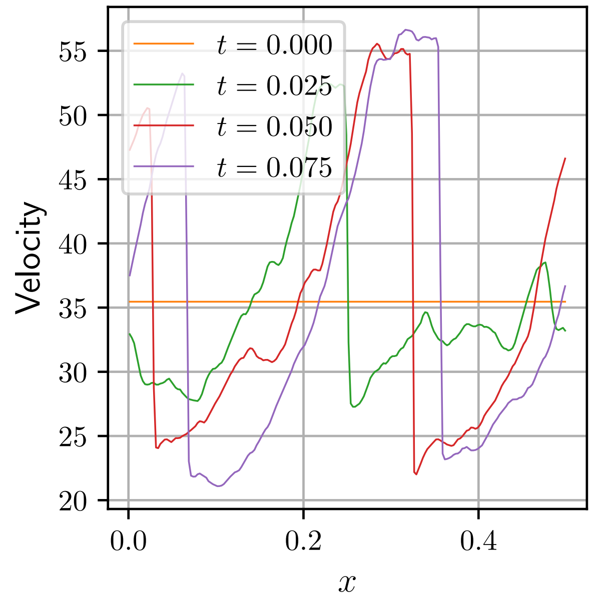

Figure 2: Velocity profiles by the dynamic model (Equations 2 and 3), periodic boundary conditions, and a uniform initial condition. Small amounts of noise yield a stop-and-go pattern.

In this talk, we will discuss choosing parameters and related difficulties. We will present numerical results of equilibrium as well dynamic stochastic models, the latter primarily using a modification of the second order model proposed in [11]

$$g(\rho) = \frac{h(\rho)}{\tau^h}\,, \quad h(\rho) = s(\rho) + \ell_c\,, \quad s(\rho) = \frac{s_{\min}}{\rho/\rho_{\text{jam}}} \tag{4}$$

here $h(\rho)$ is vehicle headway, $s(\rho)$ the unoccupied space between vehicles, ℓ_c the mean vehicle length, and $\rho_{\text{jam}} = (s_{\min} + \ell_c)^{-1}$ is the maximum density.

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