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Shock capturing and limiting in the Active Flux method

Yifan Bai *   Philip Roe†

1 Introduction

The numerics of hyperbolic problems must contend with Godunov's result that high-order methods with fixed coefficients will always produce nonphysical oscillations as soon as shockwaves or other discontinuities appear. This implies that if we wish to attack discontinuous problems with high-order methods then such a method must be in some form nonlinear. Unfortunately, there is no indication of the form that the nonlinearity should take, and more than one form has been tried.

The two best-known approaches are Flux-corrected Transport (FCT) [1] and the Monotone Upwind Scheme for Conservation Laws (MUSCL) [2] working on rather different lines. FCT is based on the practical observation that many defects in CFD are signalled by the appearance of oscillations in the solution, so a principle is adopted that no new extrema will be permitted unless they also appear in a low-order solution that is believed to be "safe" in some sense. MUSCL is based on reconstructing gradients inside each cell, and using these to enhance the Godunov method of finding fluxes by solving Riemann problems at each interface. Without any gradients, the Godunov scheme is "safe" in that it avoids vacuum conditions, and is optimal among one-dimensional entropy-satisfying schemes [3].

Neither method is safe from criticism. FCT has little supporting theory. Extrema in one set of variables (physical) may not imply extrema in another set of variables (characteristic). The FCT method seems to have found its niche for complex multidimensional multi-physics applications where there is little theory to appeal to, and the need is to do something without incurring great expense. The MUSCL scheme has more theoretical support but only in terms of one-dimensional concepts such as Total-Variation Diminishing (TVD) or Maximum-Principle Preserving (MPP). Both of these are founded on the result that holds for scalar conservation laws; the conserved variable remains constant along any characteristic line until it disappears into a shockwave. In one dimension such a result holds also for the Riemann invariants in a simple wave, but there is no such rule in more complex situations, especially in higher dimensions. Two very important applications of shockwaves, to give one peaceful example and one militaristic, are *lithotripsy*, the removal of kidney stones by focused ultrasound, and the ignition of a nuclear warhead by conventional high explosive. [1]

The second author has argued [6] that Godunov-type methods, despite their many successes, are limited in their potential. This is because they are based exclusively on one-dimensional wave propagation (even when such waves are allowed to propagate obliquely to the grid), and also because they introduce unnecessary discontinuities into the reconstruction process. From this viewpoint we need a method, reasonably sound and not too expensive, that can be applied to continuous data on unstructured grids, in situations that do not admit any "maximum principle". For example, in the compressible Euler equations, there is hyperbolic behavior along streamlines and also, in the two-dimensional steady supersonic case, along Mach lines. Such behavior does admit a maximum principle, but no principle applies in unsteady cases. Any new analysis ought to begin by clarifying alternative objectives, but these do not come readily to mind, although some authors have "invented" maximum principles to constrain undesirable behavior. What we can do is separate the issues that we know how to handle from those involving greater uncertainty. We can then experiment on the latter.

In the Active flux method, the fluxes are computed by separately allowing for advective behavior (which admits a maximum principle) and acoustic behavior (which does not). We concentrate here on the acoustic behavior, solving the acoustic equations (which omit the advective terms) in a nonlinear version sometimes called the p-system;

(1) $$\partial_t \rho + \nabla \cdot (\mathbf{v}) = 0, \qquad \partial_t(\mathbf{v}) + \nabla(\rho^\gamma) = 0$$

*Graduate Student, Department of Aerospace Engineering, University of Michigan, MI 48109, USA
†Emeritus Professor, Department of Aerospace Engineering, University of Michigan, MI 48109, USA

[1] It was pointed out by van Leer [4] that an earlier version of MUSCL was the subject of an unpublished TsAGI report by Vladimir Kolgan in 1972. Due to this recognition, a translation of Kolgan's work was published in *JCP* in 2011[5]. Not quite parallel to this is that the work of the second author, on a method similar to MUSCL in the late 1970s, was unpublished due to security restrictions.

2 Base Scheme

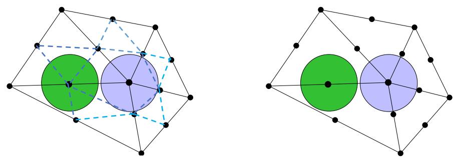


Figure 1: A cluster of elements. The solution is defined on the right by a quadratic interpolant on each element and on the left by a piecewise linear interpolant on each subelement. The discs mark domains of dependence for edge and vertex nodes.

The third-order Active Flux method for linear acoustics (which assumes constant density) was given by Fan and Roe[7]. We summarize this, noting that the nonlinear problems treated here simply assume constant coefficients around each node. Point values are available at the vertex nodes and edge midpoint nodes of every element. The exact update of each nodal value is found from the following generalizations of Poisson's formula [8].

$$\begin{aligned} \rho(\mathbf{x},t) &= \rho(\mathbf{x},0) - ctM_{ct}\{\nabla\cdot\mathbf{v}(\mathbf{x},0)\} + \int_0^{ct} rM_r\{\nabla^2\rho\}dr \\ \mathbf{v}(\mathbf{x},t) &= \mathbf{v}(\mathbf{x},0) - ctM_{ct}\{\nabla p(\mathbf{x},0)\} + \int_0^{ct} rM_r\{\nabla\nabla\cdot\mathbf{v}\}dr \end{aligned} \tag{2}$$

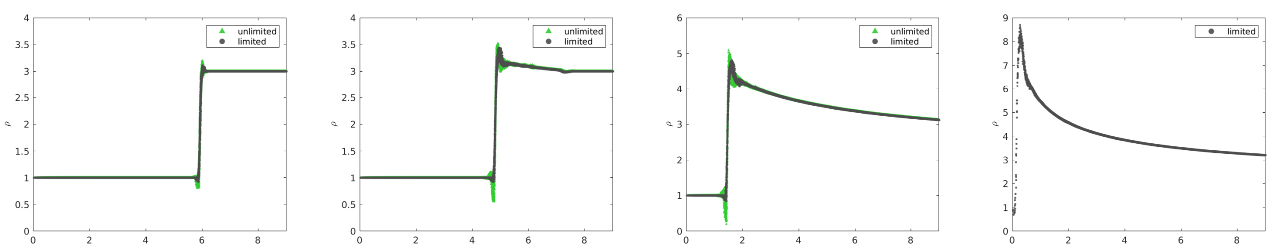
where M_{ct} is the spherical mean taken over a sphere of radius equal to ct. These are the full three-dimensional formulas, here adapted to two dimensions with integrals over the Mach discs (Fig.1). These formulas are not expensive. The high-order scheme evaluates them from a quadratic interpolation over the complete element. The low-order scheme evaluates them from linear interpolants over each subelement. The spherical means supply information from which wave directions might be deduced. If these formulas are fed with data that is one-dimensional in any direction then they supply characteristic relations in that direction.

3 Limiting and Results

As usual in FCT, we constrain the high-order predictions with the low-order ones. The high order update at a node predicted with Eqn. 2 is not allowed to exceed the minimum or the maximum of the low order predictions in its neighborhood. We define the neighborhood to be the circle with radius ct. The extrema are found by interpolations with the linear bases on the subelements.

Our test problem, in a square domain with an unstructured grid, has initial data uniform inside a circle of radius r_0 $(p \equiv 1, \mathbf{v} \equiv 0)$ and non-uniform but stationary $(p = 3, \mathbf{v} = \sqrt{8}r_0/r)$ outside that circle. Across that circle the Hugoniot conditions are satisfied. Numerical data imposes a transition over about four cells. The exact solution is a cylindrical shockwave propagating inward that grows in strength. A spherical version of this problem has been studied intensively using blast wave theory, beginning with [9].

Fig 2(a) shows a solution at an early time. Oscillations around the captured shock are reduced but not completely eliminated by imposing the limiter. At time 3.0 (Fig 2(c)) the shock is nearing the origin; its location is very crisply defined. We believe this is a result of having the propagation direction precisely determined by the spherical means. The oscillations at the foot of the shock are almost eliminated, but a steep curvature remains at the top, which may be genuine. At time 3.74, the shock is reduced to a width of about two elements. The unlimited shock has already blown up, but in the limited solution the shock (Figs 2(d),3) still shows remarkable coherence and symmetry. This bodes well for studying the important issue of stability in collapsing shocks.



(a) $t = 0.027$ (b) $t = 0.806$ (c) $t = 3.0$ (d) $t = 3.74$

Figure 2: Density solutions for a shock wave propagating inward. Note the changes of vertical scale.

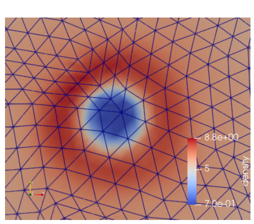


Figure 3: Density at $t = 3.74$. Although under-resolved, the shock retains impressive symmetry.

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