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Weak convergence of the finite volume method for hyperbolic systems on general meshes

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In this work we consider linear first order hyperbolic systems of the form

$$\partial_t U + \sum_k A_k \partial_k U = 0, \tag{1}$$

on $\mathbb{R}^d$ and investigate first order finite volume approximations of (1) for general initial data $U(x,0) \in L^2(\mathbb{R}^d)$ on general meshes.

Proofs of convergence of finite volume methods for hyperbolic systems on unstructured meshes generally assume some smoothness of the solution (at least H^1 with compact support in [5], H^2 in [1], H^s with $s \in]0,1]$ in [3]), and are restricted to the classical "upwind scheme" ([5, 1, 3]).
However for general possibly discontinuous initial data $U(x,0) \in L^2(\mathbb{R}^d)$ and non smooth solutions, one usually relies on compactness methods. Unlike [2] who use compactness methods in L^∞ and L^1 for scalar conservation laws, [4] uses L^2 compactness methods, in the context of hyperbolic systems. Using the Banach-Alaoglu theorem, [4] proves the weak convergence of nonlinear finite volume schemes for a general L^2 initial data provided the mesh sequence satisfies a smoothness property.

The mesh smoothness assumption in [4] is required to prove the strong convergence of a sequence of discrete test functions gradients as the mesh is refined (lemma 4 in [4]). We propose a new approach, based on the Riesz-Fréchet-Kolmogorov compactness theorem to prove the strong convergence of discrete gradient and therefore extend the result of [4] to general meshes.
Our new convergence result for L^2 initial data on general meshes is therefore based on two compactness theorem : Banach-Alaoglu theorem for the weak convergence of finite volume approximations, and Riesz-Fréchet-Kolmogorov theorem for the strong convergence of discrete test functions gradients. We believe this appoach can find applications in the finite volume approximation of elliptic equations with a two point flux scheme.

References

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