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Observer-based data assimilation for isothermal gas transport using distributed measurements

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For an efficient operation of gas pipes it is essential to know the current state of the gas in the pipes. In order to reconstruct the system state we set up an observer system that is based on the isothermal Euler equations, where distributed measurements of one of the fields mass density, velocity or mass flow are inserted into the observer through suitable source terms. Our main goal is to show exponential synchronization of the observer towards the original system state.

We model the gas transport through gas pipes by the isothermal Euler equations, which can be written in the Hamiltonian formulation

$$\begin{aligned} \partial_t \rho + \partial_x(\delta_v \mathcal{H}(\rho, v)) &= 0, \\ \partial_t v + \partial_x\left(\delta_\rho \mathcal{H}(\rho, v)\right) &= -\gamma |v| v \end{aligned} \tag{1}$$

for $0 < x < \ell,\ t > 0$ with smooth and strictly convex pressure potential $P = P(\rho)$ and associated energy functional

$$\mathcal{H}(\rho, v) := \int_0^\ell \left(\tfrac{1}{2}\rho v^2 + P(\rho)\right)\, dx,$$

whose variational derivatives are given by $\delta_\rho \mathcal{H}(\rho, v) = \frac{1}{2}v^2 + P'(\rho)$, $\delta_v \mathcal{H}(\rho, v) = \rho v$. Since the relevant scales for gas transport are long temporal and spatial scales, we consider the case of low Mach numbers, i.e. small (subsonic) velocities. The stability of the equations (1) in this regime is investigated in [1]. There, the main tool is controlling the growth of the relative energy

$$\mathcal{H}(\hat{u}|u) := \mathcal{H}(\hat{u}) - \mathcal{H}(u) - \mathcal{H}'(u)(\hat{u} - u),$$

between two solutions $u := (\rho, v)$ and $\hat{u} := (\hat{\rho}, \hat{v})$, which was introduced for hyperbolic conservation laws in [2]. Its application to a variety of thermo-mechanical theories of fluids employing a Hamiltonian structure similar to (1) can be found in [3]. Let us note that for classical, subsonic solutions of (1) that are bounded away from vacuum the relative energy is equivalent to $\|u - \hat{u}\|^2_{L^2(0,\ell)}$, which means in particular, that the relative energy can be used in order to measure the distance between two solutions. Then, exploiting the Hamiltonian structure of the equations, estimating the time derivative of the relative energy and applying a Gronwall Lemma allows to show stability of solutions of (1) with respect to parameters and initial data.

Based on the techniques that are used to show stability, we investigate the convergence for long times of an observer for distributed measurements to the original solution. Here, the dynamics of the observer are governed by the system

$$\begin{aligned} \partial_t \hat{\rho} + \partial_x(\delta_v \mathcal{H}(\hat{\rho}, \hat{v})) &= R_\rho, \\ \partial_t \hat{v} + \partial_x\left(\delta_\rho \mathcal{H}(\hat{\rho}, \hat{v})\right) &= -\gamma |\hat{v}| \hat{v} + R_v \end{aligned} \tag{2}$$

for $0 < x < \ell,\ t > 0$ with boundary conditions

$$m|_{x=0}, \hat{m}|_{x=0} = m_{\partial,0}, \qquad h|_{x=\ell}, \hat{h}|_{x=\ell} = h_{\partial,\ell} \tag{3}$$

for the mass flow $m := \rho v$ and the total specific enthalpy $h := \frac{1}{2}v^2 + P'(\rho)$. In general, (1) and (2) have different initial data, since usually only an approximation of the initial data of the exact system is known. The source terms R_ρ, R_v are of Luenberger

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type, e.g.,

$$R_\rho = 0, \qquad R_v = \mu(v - \hat{v}), \qquad \mu > 0$$

for measurements of the velocity v. A similar approach has been used in [4], but there the data assimilation is performed on the kinetic level, which does not allow a direct application of standard simulation schemes for gas transport.

Now, our goal is to prove exponential convergence of the solution of the observer system (2) to solutions of (1). It would be desirable to show this convergence in a very general setting, however, at the moment, we can show synchronization provided that (1) and (2) admit Lipschitz continuous solutions that are subsonic and bounded away from vacuum, the solution of the original system (1) is sufficiently close to a stationary state and the velocities in both systems stay sufficiently small. This assumptions are reasonable in practice, since in the usual operational range of gas pipes the gas velocities are small and only slow changes in the gas flow occur, and the bounds for the original solution can be verified theoretically, see [5]. In this setting, we use a modification of the relative energy framework that is inspired by the extension of the energy that was used in [6] to study convergence of wave equations to steady states in order to show the following theorem:

THEOREM 1 *Let $u = (\rho, v)$ and $\hat{u} = (\hat{\rho}, \hat{v})$ be Lipschitz continuous solutions of* (1) *and* (2)*, respectively, with boundary conditions* (3)*. Assume that u and $\hat{u}$ are subsonic and bounded away from vacuum, i.e., there exist constants $0 < \underline{\rho} < \bar{\rho} < \infty$ and $0 \leq \tilde{v} < \infty$ such that*

$$0 < \underline{\rho} \leq \rho(t,x), \hat{\rho}(t,x) \leq \bar{\rho}, \qquad -\tilde{v} \leq v(t,x), \hat{v}(t,x) \leq \tilde{v}$$

and

$$\rho P''(\rho) \geq 4|\tilde{v}|^2 \quad \forall\, \underline{\rho} \leq \rho \leq \overline{\rho}$$

for all $0 \leq x \leq \ell$ and $0 \leq t \leq t_{\max}$.

Then there exist constants $\mu,\ C_t,\ \bar{v} > 0$ such that for solutions u, $\hat{u}$ that satisfy

$$\|\partial_t \rho\|_{L^\infty} + \|\partial_t v\|_{L^\infty} \leq C_t$$

and

$$|v|, |\hat{v}| \leq \bar{v}$$

there exist constants $C, \tilde{C} > 0$ such that

$$\|u(t,\cdot) - \hat{u}(t,\cdot)\|^2_{L^2(0,\ell)} \leq C\|u_0 - \hat{u}_0\|^2_{L^2(0,\ell)} \mathrm{e}^{-\tilde{C}t}.$$

One interesting feature revealed by our analysis is that synchronization is not necessarily made faster by increasing the value of μ and we can, indeed, depending on the value of $\bar{v}$, only prove synchronization for a certain range of values of μ.

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