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A semi-implicit method for Saint-Venant-Exner systems with solid transport discharge formulae including gravitational terms

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In this work the Saint-Venant-Exner system is considered, which is defined by the following set of equations,

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(q^2/h\right) + gh\partial_x (h+b) = \mathcal{F}, \\ \partial_t b + \partial_x Q_b = 0, \end{cases} \tag{1}$$

where h is the height of the fluid layer, h_m is the thickness of the movable bed and b the total height of the sediment layer, see Figure 1. $\mathcal{F}$ denotes the drag term. $q = h\,u$ is the fluid discharge, being u the velocity of the fluid. Finally, Q_b denotes the solid transport discharge.

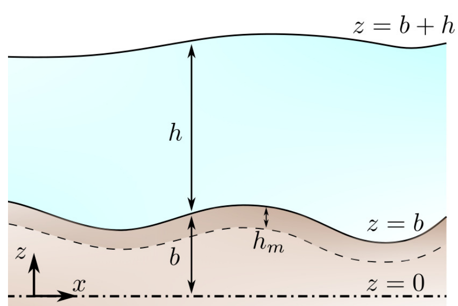

Figure 1: Domain and notation

A general formulation of the solid transport discharge can be written as follows,

$$Q_b = h_m v_b \sqrt{(1/r-1)gd_s},$$

where v_b is the dimensionless averaged velocity of the sediment layer, d_s is the grain diameter and r the aspect ration between the fluid and solid particles densities. This velocity is defined in terms of the effective shear stress (τ_{eff}) as follows

$$v_b = sgn(\tau_{\text{eff}})(\sqrt{\theta_{\text{eff}}} - \sqrt{\theta_c})_+, \quad \text{with} \quad \theta_{\text{eff}} = \frac{|\tau_{\text{eff}}|/\rho}{(1/r-1)gd_s},$$

where τ_{eff} must be properly defined. Here, following [4], we consider

$$\frac{\tau_{\text{eff}}}{\rho} = \frac{\tau}{\rho} - \frac{\vartheta g d_s}{r}\partial_x (rh+b), \quad \text{with} \quad \frac{\tau}{\rho} = C|u|u, \quad \vartheta = \frac{\theta_c}{\tan\delta},$$

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being δ the repose angle of the material and θ_c the critical Shield stress. Typical values of ϑ and θ_c are, $\vartheta = 0.1$ and $\theta_c = 0.047$, what implies that $\theta_c/\vartheta \approx \tan 25^\circ$ as proposed by Fredsœ in [5]. Other values have been also suggested (see [2] and references therein).

In order to set the definition of h_m we consider the case of a quasi-uniform regime, where h_m is defined by a closed formula as a function of the erosion-deposition rates (see [4] and reference therein for further details). A possibility is to define

$$h_m = \frac{K_e d_s}{K_d\,(1-\varphi)}\,(\theta_{\text{eff}} - \theta_c)_+\,,$$

with K_e, K_d constant parameters related to the erosion-deposition effects. Let us remark that this linear relation between the thickness of the movable bed and the shear stress has been also observed experimentally by Fernández-Luque and Van Beek (see [3]). This relation is used in the deduction of some of the most known classical formulae for the solid transport discharge.

By using this definition of h_m we obtain the following formula of the solid transport discharge

$$\frac{Q_b}{Q} = \text{sgn}\,(\tau_{\text{eff}})\,\frac{K_e}{K_d\,(1-\varphi)}\,(\theta_{\text{eff}} - \theta_c)_+\,(\sqrt{\theta_{\text{eff}}} - \sqrt{\theta_c})_+ \qquad \text{where} \quad Q = d_s\sqrt{(1/r-1)gd_s}.$$

Note that due to the definition of θ_{eff}, the last equation of system (1) is a parabolic-degenerated non-linear equation (see [6]). In this work we present an efficient semi-implicit method (see [1]) for Saint-Venant-Exner models with gravitational effects under subcritical regimes.

Several numerical tests will be presented, where we focus on the application of a generalization of the Ashida-Michieu model. Analytical steady states solutions (both lake at rest and $|u| > 0$) are deduced an approximated with the proposed scheme. We can observe that in all the presented tests, the proposed method allows us to notably reduce the computational effort without a significant loss of accuracy.

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