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Splitting methods for a generalized blood flow equation model applied on networks

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We consider a model for blood flow in straight and deformable tubes [1], namely

$$\partial_t A + \partial_x q = 0 \, ,$$
$$\partial_t q + \partial_x \left(\alpha \frac{q^2}{A} \right) + \frac{A}{\rho} \partial_x p = 0 \, , \tag{1}$$

where t is time, x is distance, A is the cross-sectional area, q is the volumetric flow rate, ρ is the blood density, α is a momentum correction factor given by

$$\alpha = \frac{\int_S s^2 \, d\sigma}{A} \, , \tag{2}$$

with $s(y, z)$ being a velocity profile function, with y and z representing the coordinates of the plane perpendicular to x, while S is the cross-section of the vessel, always perpendicular to x. Moreover, $p(A; Z)$ is the blood pressure, assumed here to be an algebraic function of cross-sectional area A and of a set of parameters $Z(x) \in \mathbb{R}^L$, with L parameters that are allowed to be discontinuous (or at least vary in space). Such parameters include geometrical and mechanical properties of the vessel and while discontinuities in a single vessel might be considered uncommon, jumps in parameters naturally arise when multiple one-dimensional domains need to be coupled at junctions/bifurcations.

In order to consider (discontinuous) variations of geometrical and mechanical properties we reformulate the original problem as proposed in [2], namely

$$\partial_t \mathbf{Q} + \mathbf{A}(\mathbf{Q}) \partial_x \mathbf{Q} = \mathbf{0} \, , \tag{3}$$

where

$$\mathbf{Q} = [q_1, q_2, q_3, \ldots, q_{2+L}]^T = [A, Au, \zeta_1, \ldots \zeta_L]^T \tag{4}$$

and

$$\mathbf{A}(\mathbf{Q}) = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ c^2 - \alpha u^2 & 2\alpha u & \frac{A}{\rho}\frac{\partial p}{\partial \zeta_1} & \frac{A}{\rho}\frac{\partial p}{\partial \zeta_2} & \ldots & \frac{A}{\rho}\frac{\partial p}{\partial \zeta_L} \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix} . \tag{5}$$

We first provide an in-depth analysis of system (3) and its related Riemann problem in the sub-critical regime. In doing so we assume a general tube law (in compressible media this would be a general equation of state) and explore which properties such

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tube law must satisfy to guarantee that the PDE system possesses specific, desirable mathematical properties. In our analysis we will emphasize how knowledge on this basic problem turns out to be useful in the definition of coupling conditions when networks of vessels are considered. While a common assumption in the literature is $\alpha = 1$, we concentrate here in the case when $\alpha \geq 1$. Such generalization has a profound impact on the solution strategy for the Riemann problem; in particular, no closed-form expressions for generalized Riemann invariants are available and one has to resort to computing integral curves numerically. Moreover, since Riemann invariants play a crucial role in determining coupling and boundary conditions for blood flow in networks, the assumption $\alpha \geq 1$ has also an import impact on the computational efficiency with which such conditions can be computed.

This problem turns out to be an excellent example in which flux vector splitting becomes crucial. In fact, by adapting [3] the splitting approach proposed in [4] we can deal separately with a pressure and an advection system. The pressure system does not contain terms involving α and allows for straightforward and efficient computation of Riemann invariants and, consequently, related coupling and boundary conditions. The discretization of the pressure system is performed here through Godunov-type numerical fluctuations [5] obtained with the Dumbser-Osher-Toro approach [6].

We will assess the accuracy of the splitting approach when used to approximate solutions for Riemann problems, as well as when applied in the context of blood flow simulations in large networks of vessels. In this last case efficiency improvements will be assessed.

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