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Large time behavior of finite difference schemes for the transport equation

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We introduce a time step Δt, a space step Δx, $\lambda := \frac{\Delta t}{\Delta x}$ and we consider an explicit difference scheme with constant coefficients of the transport equation such that the modified equation associated to the scheme is

$$\partial_t u + \alpha \partial_x u = \frac{\beta}{\Delta t} \Delta x^{2\mu} \partial_x^{2\mu} u,$$

where $\beta \in \mathbb{R}_+^*$ and $\mu \in \mathbb{N}^*$. This scheme can be rewritten as a system of the form

$$\left\{ \begin{array}{l} \forall n \in \mathbb{N}, \quad v_{n+1} = a * v_n, \\ \qquad v_0 \in \ell^p(\mathbb{Z}), \end{array} \right. \tag{1}$$

for some $a \in \ell^1(\mathbb{Z})$ with finite support. If we introduce $a^n := a * \cdots * a$, we are interested in studying the sequence a^n as n becomes large to understand the long time behavior of solutions of (1).

The goal of the talk will be to present [1, Theorem 1] which claims that if we introduce the function

$$\forall x \in \mathbb{R}, \quad H_{2\mu}^{\beta}(x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixu} e^{-\beta u^{2\mu}} du.$$

then, under some more assumptions on a, there exist two constants $C, c > 0$ such that

$$\forall n \in \mathbb{N}^*, \forall j \in \mathbb{Z}, \quad \left| a_j^n - \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}^{\beta}\left(\frac{j - n\lambda\alpha}{n^{\frac{1}{2\mu}}}\right) \right| \leq \frac{C}{n^{\frac{1}{\mu}}} \exp\left(-c\left(\frac{|j - n\lambda\alpha|}{n^{\frac{1}{2\mu}}}\right)^{\frac{2\mu}{2\mu-1}}\right).$$

This result falls in the domain of the study of convolution powers for sequences in $\ell^1(\mathbb{Z})$. More precisely, it can be seen as a generalization for complex valued sequences of the so-called local limit theorem in probability theory, which aims at finding asymptotic expansions of a_j^n and looking for bounds on the remainder. There have been recent developments around this question in two different directions

- In [2, 3], we can find proofs of generalized gaussian bounds on the coefficients a_j^n.
- in [3, 4], the authors find the precise expression of the asymptotic behavior of a_j^n and prove an asymptotic expansion of the form
$$a_j^n - \frac{1}{n^{\frac{1}{2\mu}}} H_{2\mu}^{\beta}\left(\frac{j - n\lambda\alpha}{n^{\frac{1}{2\mu}}}\right) \underset{n\to+\infty}{=} o\left(\frac{1}{n^{2\mu}}\right).$$

[1, Theorem 1] allows us to link those two developments, give a more precise description of asymptotic behavior of the sequence a^n, and thus have a better understanding of the large time behavior of solutions of our scheme (1).

During the talk, the objective is also get some insight on the proof of this theorem. In the articles [4, 3], the proofs mainly rely on the use of Fourier analysis to express the elements a_j^n via the Fourier series associated with a. In [1], we use an approach usually

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referred to in partial differential equations as "spatial dynamics" (just like in [2]). It aims at using the functional calculus on the Laurent operator L_a associated to a defined by

$$\forall v \in \ell^p(\mathbb{Z}), \quad L_a v := a * v \in \ell^p(\mathbb{Z}).$$

The goal is to express the temporal Green's function (here the coefficients a_j^n) with the resolvent of the operator L_a via the spatial Green's function which is the unique solution of

$$(zId - L_a)u = \delta, \quad z \in \mathbb{C} \backslash \sigma(L_a),$$

where δ is the discrete Dirac mass $\delta = (\delta_{j,0})_{j \in \mathbb{Z}}$. The proof of [1, Theorem 1] relies on a detailed analysis of the spatial Green's function with a sharp holomorphic extension and an extraction of its precise behavior near certain points of interest. It is followed by a suitable choice of contour to express the temporal Green's function with the spatial Green's function.

The talk will be ended with some insight on the link between the result we presented and the stability of discrete shock profiles for systems of conservation laws.

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