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Second order finite volume IMEX Runge-Kutta schemes for option pricing nonlinear PDEs in finance

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The goal of this article is to develop a general technique for building second order numerical schemes for financial nonlinear advection-diffusion-reaction PDEs in 1d and 2d. The technique is based in the use of finite volume implicit-explicit (IMEX) Runge-Kutta numerical methods. More precisely we present second order numerical methods for solving one-dimensional nonlinear option pricing PDEs, as well as baskets and Heston models in two spatial dimensions, with mixed derivatives. To our knowledge, this is the first time that finite volume IMEX Runge-Kutta numerical schemes have been successfully applied to the numerical solution of nonliear parabolic financial PDEs. The interplay between finite volume methods and IMEX time integrators is an extremely powerful combination in mathematical finance. On the one hand, finite volume methods are well suited to treat convective terms. In fact, it allows to build schemes with true second order convergence, even in the presence of non-regular initial or boundary conditions, which is a well-known difficulty in the financial literature. On the other hand, IMEX time integrator makes possible to overcome the tiny time steps induced by the spatial semi-discretization of the diffusive terms. This is also of paramount importance, otherwise the problem is intractable from the computational point of view. On top of that, the presented numerical schemes provide excellent finite difference approximations of the Greeks. Option pricing methods can be mainly classified into two categories, deterministic and stochastic algorithms. The majority of the first ones deal with the solution of PDEs, while the second ones employ the Monte Carlo method based on the formulation of the model in terms of expectations. More precisely, in this work we will focus in pricing options under financial models with one and two spatial variables, that can be written as a 1d or 2d advection-diffusion-reaction PDE with possible nonlinearities in the convection and/or in the reaction terms.

Standard numerical methods for solving parabolic PDEs in finance, are finite difference (FD) based, see [1, 6, 10, 9], or finite elements (FE) based, see [1, 2]. Another possibility is to use Exponentially Fitted (EF) finite differences schemes (see [6]) or Alternate Directions (ADI) finite differences methods. Nevertheless, FD or FE numerical schemes for financial PDEs face several mathematical challenges. The first one is due to the lack of smoothness of the initial conditions imposed by most of the traded option contracts. This non-smooth data originates noise in the numerical solution. As a result, the expected rate of convergence is seriously degraded. Besides, the numerical computation of derivatives of the numerical solution originates poor estimates of the Greeks with oscillations. Another mathematical challenge in the numerical solution of this kind of advection-diffusion PDEs arises when the advection term becomes larger than the diffusion one, or when we have degenerated diffusion. Under this situation, instabilities show up, because the problem becomes more hyperbolic. In order to face these problems, several techniques have been introduced. One way to overcome these instability phenomena is to avoid centered schemes and to consider some flavour of upwind discretizations; we refer the reader to [1], [5], [2]. Another approach to properly solve convection-dominated diffusion problems is to consider finite volume discretization methods. Finite volume methods naturally handle non smooth payoffs and the cases where the PDE becomes diffusion degenerated. Seminal works on applying finite volume method to option pricing problems are due to Forsyth and Zvan, see [12, 11], where vertex type finite volume methods were applied for problems in two spatial dimensions. More recently, in [4] the authors propose a second order improvement of [8] with appropriate time methods and slope limiters. Finally in [3] the authors apply a third order Kurganov-Levy scheme. All these works only deal with linear Black-Scholes PDEs in dimension one or linear Asian PDE problems in 2d. Therefore, they do not solve nonlinear problems or general two-dimensional pricing problems with mixed derivatives. Aso in these works, the

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ordinary differential equations (ODEs) obtained after semi-discretizing in space are stiff, and the authors use explicit-schemes in time. The main disadvantage of these approaches is that time steps have to be very small to maintain stability. Typically, Von Neumann stability analysis demands $\Delta t \leq (\Delta x)^2/2\eta$, being η the diffusion velocity. Pareschi and Russo proposed in [7] implicit-explicit (IMEX) Runge-Kutta method for general hyperbolic systems of conservation laws with stiff relaxation terms. The idea is to apply an implicit discretization to the source terms and an explicit one to the nonstiff term. All in all, our goal is to first semi-discretize in space 1d and 2d Black-Scholes equations with second-order finite volume methods, thus properly treating convection terms and non-smooth payoffs. Later, we propose to integrate in time the resulting system of stiff ODEs by means of the second-order IMEX Runge-Kutta time marching scheme. In this way, we will apply an implicit discretization to the diffusion (stiff) part and an explicit one to the convection and source terms (non stiff). As a result, the time step restriction will depend only on the stability of the degenerate diffusion PDE, i.e, $\Delta t \leq \Delta x/\alpha$, being α the convection velocity.

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