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High-order well-balanced numerical schemes for shallow-water systems with Coriolis terms

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The goal of this work is to develop high-order well-balanced schemes for the one-dimensional shallow-water equations with Coriolis terms. We will focus on the one-dimensional shallow-water system with Coriolis terms. To the best of our knowledge, this is the first time in the literature, that such a general technique is presented for the construction of high-order well-balanced numerical methods in the context of 1D shallow-water equations with Coriolis forces.

The system is given by:

$$\partial_t U + \partial_x f(U) = s(x, U), \quad s = s^B + s^C \tag{1}$$

where

$$U = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}, \; f(U) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{bmatrix}$$

are the conserved variables and the flux in the x direction respectively, and

$$s^B = \begin{bmatrix} 0 \\ -gh\partial_x z \\ 0 \end{bmatrix} \quad s^C = \begin{bmatrix} 0 \\ fhv \\ -fhu \end{bmatrix}$$

are the source terms due to the bottom topography and the Coriolis forces respectively. This problem fits into the more general framework of building high-order schemes for balance laws

$$\partial_t U + \partial_x f(U) = s(x, U), \qquad U \in \mathbb{R} \times \mathbb{R}^+ \to \Omega \subset \mathbb{R}^N, \quad f \in C^1(\mathbb{R}^N, \mathbb{R}^N), \quad s\colon \mathbb{R} \times \Omega \mapsto \mathbb{R}^N, \tag{2}$$

that are well-balanced, that is, they preserve, in some sense, all or certain families of stationary solutions of the system.

Several works about developing well-balanced numerical schemes for system (1) can be found in the literature. For example, in [4] the authors focused on the simulation of the geostrophic adjustment and developed 1st-order Roe schemes and some higher-order extensions, to obtain good approximations for inertial oscillations, and also studied the wave amplifications of the schemes, together with the well-balancedness properties for the steady states corresponding to $u = 0$ and $v = C$, $C \in \mathbb{R}$. In [1] the authors developed first and second-order schemes for the shallow-water equations with bottom topography and Coriolis forces that preserve the water height positivity. They used non-local potential operators to obtain a well-balanced numerical scheme up to second-order, for the steady steady states with $u = 0$. They also consider its natural extension to 2D problems. In [7] the problem of developing a fully well-balanced scheme for the one-dimensional shallow-water system with Coriolis terms is also addressed. This article describes a technique to obtain well-balanced second order schemes for the system. Other interesting papers related to well-balanced numerical methods for the shallow-water system with Coriolis are ([3], [9], [2]).

Here will follow the strategy presented in [5]. Following [4], [5], a well-balanced method can be built by considering well-balanced reconstruction operators. The critical step of this procedure is solving the following non-linear problem for each cell

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and time step:

$$
(3) \qquad \begin{cases} \partial_x f(U) = s(x, U), \\ \dfrac{1}{|I_i|} \displaystyle\int_{x_{i-1/2}}^{x_{i+1/2}} U(x)\, dx = U_i. \end{cases}
$$

After solving (3) at the volume, the obtained solution, $U_i^*(x)$ is extended to the whole stencil. In certain situations, it is not possible to obtain the exact solutions of (3). In this case, $U^*(x)$ will be replaced by a suitable approximation that will be denoted by $\widetilde{U}^*(x)$. Following [8], in this case, the semi-discrete finite volume numerical scheme is well-balanced, if the sequence of cell averages computed from the approximation $\widetilde{U}^*(x)$, and denoted by $\{U_i^*\}$, is an equilibrium of the system of the ODE system given by the semidiscrete scheme.

In this work following [5] in combination with CWENO reconstruction operators (see [6]) we are able to define arbitrary high-order exactly well-balanced numerical schemes with $u \neq 0$, and also the special case $u = 0$ will be discussed. The main difference between the cited papers with the current approach is that, because of the generality of the presented technique, it allows the building of arbitrary high-order numerical schemes, not restricted to second-order. Also, for the 2-dimensional schemes, we have a first work in [10], to be published, where we follow an extension of the numerical scheme proposed in [1]. In this and future works, we are combining local and global solvers to determine stationary solutions for the system.

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