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A framework to construct multidimensional constraint preserving and discrete well-balanced schemes

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We consider the construction of very high order and well balanced approximations for systems of balance laws.

We start by briefly addressing the question of how to define what an equilibrium is, and what well balanced scheme means. Then, as in the recent work by M. Castro, C Parés and collaborators (e.g. [1, 2] and references therein), we focus on a discrete definition of this property. In particular we will consider schemes that preserve exactly some very accurate discrete approximation of all the steady states of the given PDE system.

We will then discuss some of the work by [3, 4] and introduce the notion of global flux quadrature in one space dimension. This is essentially a local strategy to evaluate source integrals in the finite element context. At steady state this approach allows to recover a strong consistency with implicit Runge-Kutta collocation methods. This provides a clean and neat interpretation of the discrete steady states. In practice, a remarkable reduction in the approximation error of such states is always observed. Moreover, for well chosen finite element bases, one can to obtain a super-convergent behaviour, well predicted by the classical ODE theory, which further improves the preservation of any steady equilibrium. Numerical results are shown to support these claims.

We will then move to the multidimensional case, following very recent work by [5, 6]. We will focus on a particular family of linear hyperbolic balance laws, representative of well known ones as the rotational shallow water equations, or the Euler equations with gravitational effects. First, we address the fundamental question of what well balancing means in multiple space dimensions, and what we are exactly trying to approximate. We will show how to generalize the notion of global flux quadrature to Cartesian tensor finite element approximations in such a way that we recover discrete preservation of (non homogeneous) solenoidal and curl constraints. The global flux approach allows to exploit the properties of high order ODE integrators and to prove similar consistency, and possibly super-convergence, results as those obtained in one space dimensions. Numerical results show the enourmous enhancements of the approximation of analytical steady states, as well as the exact preservation of the corresponding discrete ones in two space dimensions.

References

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