On wave speed estimates for Rusanov-type schemes: monotonicity and stability

Eleuterio F. TORO*, Arturo HIDALGO†, Svetlana TOKAREVA‡

We are concerned with theoretical wave speed estimates for numerical schemes to solve hyperbolic equations. We first briefly review some recent advances reported in [1], where new theoretical estimates for bounds of the true maximal wave speeds for three specific hyperbolic systems are proposed, namely the shallow water equations, the blood flow equations and the Euler equations for ideal gases. Apart from one exception, it is also shown that all of the existing methods available to obtain wave speed estimates for HLL-type schemes [3] [4] fail to provide **bounds** for the maximal true wave speeds. The consequences of these findings are still uncertain in computational practice, and are worth exploring further.

The main part of this talk focuses on the potential consequences resulting from uncertain wave speed estimates on the properties of numerical schemes, such as monotonicity and linear stability. We perform a detailed analysis of the Rusanov scheme [2], applied to the linear advection equation in one and multiple space dimensions. Rusanov-type fluxes are often used in finite volume and discontinuous Galerkin methods for solving hyperbolic equations. The Rusanov flux [2], often called *local Lax-Friedrichs*, is the simplest upwind flux, requiring a single wave-speed estimate $\hat{s}$ to fully determine the scheme. In general, the estimation of $\hat{s}$ is subject to uncertainties. On the bases that such uncertainties can be represented by a parameter $\beta \in [\beta_{min}, \beta_{max}]$ we show that the estimate $\hat{s}(\beta)$ yields a broad family of schemes of the Rusanov type. In particular, for the linear advection equation with characteristic speed λ we set $\hat{s} = \beta\lambda$ and the Rusanov scheme reproduces several well-known methods, such as the Godunov upwind method, Lax-Wendroff, the Godunov centred method, FORCE, the FORCE-α method and the Lax-Friedrichs method. In our detailed analysis we show that the choice of the wave speed estimate $\hat{s}$ has a profound effect on the monotonicity and linear stability properties of the resulting numerical methods. The main results in one space dimension are represented in the x-t plane of Figure 1, where each wave speed corresponds to a characteristic line. The true wave speed corresponds to the Godunov upwind method. Four main regions are observed. A left region LR between the unconditionally unstable FTCS scheme and the Lax-Wendroff method; a middle region MR between the Lax-Wendroff method and the Godunov upwind method; a right region RR between the Godunov upwind method and the Lax-Friedrichs method, and finally a region UU to the right of the Lax-Friedrichs method consisting of unconditionally unstable schemes. Various well known schemes fall within the two regions of useful schemes, namely MR and RR. In this talk we fully characterise all the well-known schemes of Figure 1 in terms of monotonicity and stability in one and two space dimensions. Conjectures are also posed for **all possible schemes** in regions MR and RR for one and multiple space dimensions.

References

[1] E.F. Toro, A. Siviglia, L.O. Mueller. Bounds for wave speeds in the Riemann problem: direct theoretical estimates. *Computers and Fluids*, 209, 2020.

[2] V.V. Rusanov. Calculation of interaction of non-steady shock waves with obstacles. *J. Comput. Math. Phys. USSR*, 1: 267–279, 1961.

[3] H. Harten, P. Lax, B. van Leer. On upstream differencing and Godunov-type schemes for hyperbolic conservation laws. *SIAM Review*, 25(1): 35–61, 1983.

[4] E.F. Toro, M. Spruce, W. Speares. Restoration of the contact surface in the HLL Riemann solver. *Shock Waves*, 4: 25–34, 1994.

*University of Trento, Italy. Email: eleuterio.toro@unitn.it
†Universidad Politécnica de Madrid, Spain. Email: arturo.hidalgo@upm.es
‡Los Alamos National Laboratory, USA. Email: tokareva@lanl.gov

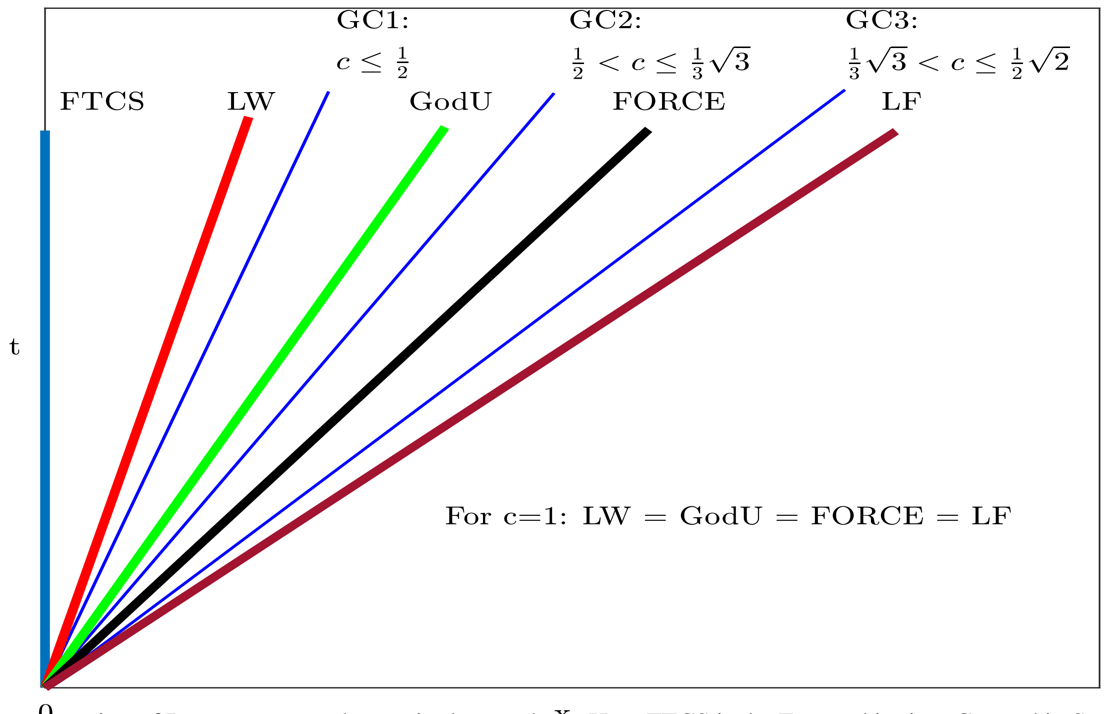

Figure 1: Representation of Rusanov-type schemes in the x-t plane. Here FTCS is the Forward in time Centred in Space scheme; LW is Lax-Wendroff; GodU is Godunov upwind; FORCE is the FORCE scheme and LF is Lax-Friedrichs. The three cases of the Godunov centred scheme are represented by GC1, GC2 and GC3. Each scheme corresponds to a particular choice of the characteristic line $x/t = \hat{s}$ emerging from the origin O. All linearly stable schemes lie in the wedge LWOLF. Large values of $\hat{s}$ are associated with more diffusive schemes, while low values of $\hat{s}$ are associated with non-monotone schemes.